



# Slow Crack Growth of Brittle Materials With Exponential Crack-Velocity Formulation— Part 1: Analysis

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# **Slow Crack Growth of Brittle Materials With Exponential Crack-Velocity Formulation—Part 1: Analysis**

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## **Summary**

Extensive slow-crack-growth (SCG) analysis was made using a primary exponential crack-velocity formulation under three widely used load configurations: constant stress rate, constant stress, and cyclic stress. Although the use of the exponential formulation in determining SCG parameters of a material requires somewhat inconvenient numerical procedures, the resulting solutions presented gave almost the same degree of simplicity in both data analysis and experiments as did the power-law formulation. However, the fact that the inert strength of a material should be known in advance to determine the corresponding SCG parameters was a major drawback of the exponential formulation as compared with the power-law formulation.

## **Introduction**

Advanced ceramics are candidate materials for structural applications in advanced heat engines and heat recovery systems. The major limitation of these materials in hostile environments, particularly at elevated temperatures, is slow-crack-growth (SCG)-associated failure, where slow crack growth of inherent defects or flaws can occur until a critical size for catastrophic failure is reached. To ensure accurate life prediction of ceramic components, it is important to accurately evaluate the SCG parameters of a material with specified loading and environmental conditions.

Life prediction (or SCG) parameters of a material depend on what type of crack-velocity formulation is used to determine them. The power-law crack-velocity formulation has been used for several decades to describe SCG behavior of a variety of brittle materials ranging from glass and glass ceramics to advanced structural ceramics. The main advantage of the power-law formulation over other crack-velocity formulations lies in the simplicity in its mathematical expression for lifetime analysis. It has also been observed that the power-law formulation has described adequately the SCG behavior of many brittle materials. Because of these merits, the power-law formulation has been used in two recent ASTM test standards (refs. 1 and 2) to determine SCG parameters of advanced ceramics in constant stress rate testing at both ambient and elevated temperatures. Alternative crack-velocity formulations take exponential forms to account for the influence of other phenomena (such as a corrosion reaction, diffusion control, thermal activation, etc.). However, these exponential forms generally do not result in simple mathematical expressions of life prediction formulation, although the forms might better represent the actual SCG behavior of some materials. Because of this mathematical inconvenience, the exponential crack-velocity formulation has rarely been used for brittle materials as a means of life prediction methodology in testing or analysis.

In this report, the exponential crack-velocity formulation was analyzed to achieve a more convenient and simplified life prediction analysis compared with the previous exponential crack-velocity-based analyses. The numerical analysis presented here was made for three widely utilized load configurations: constant stress rate (dynamic fatigue), constant stress (static fatigue or stress rupture), and cyclic stress (cyclic fatigue). The resulting analysis obtained with the exponential formulation was compared with that of the power-law formulation to assess which would yield a better life prediction methodology in terms of accuracy and convenience in testing and analysis. To the authors' best knowledge, no analytical study on slow crack growth has been done previously using the exponential formulation under cyclic loading. In the following reports (parts 2 and 3 of this series) the merits and limitations of the exponential formulation will be further described in detail using a variety of SCG data determined for many glasses and advanced ceramics at both ambient and elevated temperatures.

All symbols used in this report are listed in the appendix.

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## Theoretical Background

### Power-Law Formulation

The widely utilized empirical power-law crack-velocity term for above the fatigue limit is expressed in the form (ref. 3)

$$v = \frac{da}{dt} = A \left( \frac{K_I}{K_{IC}} \right)^n \quad (1)$$

where

$v$	crack velocity
$a$	crack size
$t$	time
$K_I$	mode I stress intensity factor
$K_{IC}$	mode I critical stress intensity factor (or fracture toughness)
$A, n$	material- and environment-dependent SCG parameters

Typically, SCG testing to determine related SCG parameters is performed by applying constant stress rate, constant stress, or cyclic stress loading to ground-test specimens. Constant stress rate testing determines strength as a function of applied stress rate, whereas constant stress and cyclic stress testing measure time to failure as a function of applied stress. The strength in constant stress rate and the time to failure in constant stress and cyclic stress tests can be analytically derived to give the following relations (refs. 4 and 5):

$$\sigma_f = D_d \dot{\sigma}^{1/(n+1)} \quad (2)$$

$$t_{fs} = D_s \sigma^{-n} \quad (3)$$

$$t_{fc} = D_c \sigma_{\max}^{-n} \quad (4)$$

where  $\sigma_f$  is the fracture stress corresponding to the applied stress rate  $\dot{\sigma}$  in constant stress rate testing,  $t_{fs}$  is the time to failure subjected to a constant applied stress  $\sigma$  in constant stress testing, and  $t_{fc}$  is the time to failure subjected to cyclic loading with a maximum stress  $\sigma_{\max}$  in cyclic stress testing. The parameters represented by  $D$ 's are expressed as follows (refs. 4 and 5):

$$D_d = \left[ B(n+1) S_i^{n-2} \right]^{1/(n+1)} \quad (5)$$

$$D_s = B S_i^{n-2} \quad (6)$$

$$D_c = \frac{(B S_i^{n-2})}{\left\{ \frac{1}{\tau} \int_0^{\tau} [f(t)]^n dt \right\}} \quad (7)$$

where  $B = 2K_{IC}/[AY^2(n-2)]$  where  $Y$  is the crack geometry factor in the relation  $K_I = Y\sigma a^{1/2}$ ;  $S_i$  is the inert strength at which no slow crack growth occurs; the function  $f(t)$  is a periodic function in cyclic loading specified in  $\sigma(t) = \sigma_{\max} f(t)$  in a range of  $0 \leq f(t) \leq 1$ ; and  $\tau$  is the period. The SCG parameters  $n$  and  $D$  (and  $B$  or  $A$ ) can be obtained by a linear regression analysis with experimental data in conjunction with the corresponding equation, either (2), (3), or (4), depending on the type of loading. Hence, it is straightforward to determine SCG parameters  $n$  and  $D$  by least-squares fitting of the data, which is the most advantageous feature of the power-law crack-velocity formulation. This convenience and merit in mathematical simplicity in addition to the use of routine test techniques have led for several decades to the almost exclusive use of the power-law crack-velocity formulation in life prediction analysis and testing for many brittle materials over a wide range of temperatures.

### Exponential Formulation

Fracture-mechanics-based modeling typically offers a framework in which lifing can be made. However, long-term life prediction is sensitive to the relation between the slow crack velocity and the stress intensity factor, which depends on many factors itself. As a result, several different exponential crack-velocity formulations that have been previously proposed are based on these other factors, which include the presence of a chemically assisted corrosion reaction (ref. 6), diffusion-controlled stress rupture (ref. 7), a thermally activated process (ref. 8), a chemical reaction with constant crack-tip configuration (ref. 9), kinetic crack growth (ref. 10), and others (ref. 11). The generalized exponential crack-velocity forms thus proposed are

$$v = A \exp \left[ n \left( \frac{K_I}{K_{IC}} \right) \right] \quad (8)$$

$$v = A \left( \frac{K_I}{K_{IC}} \right) \exp \left[ n \left( \frac{K_I}{K_{IC}} \right) \right] \quad (9)$$

$$v = A \left( \frac{K_{IC}}{K_I} \right) \exp \left[ n \left( \frac{K_I}{K_{IC}} \right) \right] \quad (10)$$

$$v = A \exp \left[ n \left( \frac{K_I}{K_{IC}} \right)^2 \right] \quad (11)$$

$$v = A \left( \frac{K_I}{K_{IC}} \right) \exp \left[ n \left( \frac{K_I}{K_{IC}} \right)^2 \right] \quad (12)$$

where  $A$  and  $n$  are SCG parameters and are different from those used in the power-law formulation. Unlike the power-law crack-velocity formulation, the exponential crack-velocity forms do not yield simple, analytical expressions of either the resulting strength as a function of applied stress rate in constant stress rate testing or of the resulting time to failure as a function of applied stress in constant stress testing or maximum applied stress in cyclic stress testing. Several attempts have been made under both constant stress rate and constant stress loading to obtain corresponding lifetime expressions through numerical integration incorporating linear (refs. 12 and 13) or nonlinear (ref. 14) regression analysis. However, this approach still involves complexity in regression technique as compared with the simple least-squares approach in the power-law formulation.

Slow-crack-growth analyses of three load configurations of constant stress rate, constant stress, and cyclic stress were made in this section to obtain simpler, representative equations through numerical solution, which in turn makes the use of regression analysis easier to determine corresponding SCG parameters comparable to the case of the power-law formulation. Trantina (ref. 12) used the exponential crack-velocity forms (eqs. (8) to (10)) to determine the approximate time-to-failure equations under constant stress rate and constant stress loading and showed that the coefficients of the exponential equations (8) to (10) were insignificant except at very low fracture stress. Ritter et al. (ref. 13) used the exponential form (eq. (8)) to determine the relation of strength versus stress rate in constant stress rate loading via a numerical method for indentation cracks that possess a residual stress field around the indent. For the purpose of simplicity and generalization, equation (8) was chosen for the present analysis. This equation was taken from Wiederhorn and Bolz (ref. 9), who modified the original Hillig and Charles (ref. 6) exponential formulation. An additional analysis using other crack-velocity forms (eqs. (9) to (11)) was also made and the results will be discussed in the section Other Exponential Formulations.

To minimize the number of parameters to be specified (such as  $A$ ,  $a$ ,  $\sigma$ ,  $S_i$ ,  $K_{IC}$ , and  $t$ ), it is convenient to use a normalized scheme, as used previously for the power-law velocity formulation (refs. 15 to 17):

$$K^* = \frac{K_I}{K_{IC}}; \quad T^* = \frac{A}{a_i} t; \quad C^* = \frac{a}{a_i}; \quad \sigma^* = \frac{\sigma}{S_i}; \quad \dot{\sigma}^* = \frac{\dot{\sigma}}{T^*}; \quad \sigma_{\max}^* = \frac{\sigma_{\max}}{S_i} \quad (13)$$

where

$K^*$  stress intensity factor (SIF)  
 $T^*$  time  
 $C^*$  crack size



$\sigma^*$  applied stress  
 $\dot{\sigma}^*$  applied stress rate  
 $\sigma_{\max}^*$  maximum applied stress

in cyclic loading, and  $a_i$  is the critical crack size in the inert condition or is the initial crack size. Using these variables, the exponential crack-velocity equation (8) can be normalized as follows:

$$\frac{dC^*}{dT^*} = e^{nK^*} \quad (14)$$

The corresponding normalized SIF  $K^*$  is expressed in load configurations of constant stress rate, constant stress, and cyclic sinusoidal stress, respectively, as

$$K^* = \dot{\sigma}^* T^* (C^*)^{1/2} \quad (15)$$

$$K^* = \sigma^* (C^*)^{1/2} \quad (16)$$

$$K^* = \left\{ \frac{1+R}{2} + \frac{1-R}{2} \sin \left[ \left( \frac{\omega a_i}{A} \right) T^* \right] \right\} \sigma_{\max}^* (C^*)^{1/2} \quad (17)$$

where  $R$  is the stress (or load) ratio, defined as  $R = \sigma_{\min}/\sigma_{\max}$ , in which  $\sigma_{\min}$  and  $\sigma_{\max}$  are the minimum and maximum applied stresses, respectively, applied in cyclic loading; and  $\omega$  is the angular velocity. As typical for ceramics, the crack size at instability in either an inert or fatigue environment was assumed to be small compared with the body of the specimens or components (i.e., an infinite-body assumption). Differential equation (14) was solved numerically using a fourth-order Runge-Kutta method for each respective loading configuration. The initial condition was  $C^* = 1.0$  at  $T^* = 0$ , and the instability conditions were  $K^* = 1.0$  and  $dK^*/dC^* > 0$ . In cyclic loading, the frequency was taken as arbitrary values of  $\omega a_i/A \geq 10^8$ , depending on the values of maximum applied stress and  $n$ . The effect of frequency on the solution is discussed in the section Cyclic Stress Loading.

## Results of Numerical Analysis

### Constant Stress Rate Loading

The results of the numerical solution of normalized fracture stress (strength)  $\sigma_f^*$  as a function of normalized stress rate  $\dot{\sigma}^*$  are shown in figure 1 for values of  $n$  ranging from 5 to 100. As seen in the figure, for a given  $n$ , strength decreases with decreasing stress rate and represents the susceptibility to slow crack growth. The rate of decrease in strength with decreasing stress rate becomes more significant with lower  $n$  values, analogous to the case for the power-law formulation, indicating that the lower  $n$  gives rise to the greater SCG susceptibility and vice versa. The strength approaches its corresponding inert strength as stress rate increases to a certain value at which no slow crack growth occurs. Likewise, the strength converges close to zero as the stress rate approaches  $\ln \dot{\sigma}^* = 0$ . A linear relationship between  $\sigma_f^*$  and  $\ln \dot{\sigma}^*$  holds for most of  $n$  values within the range of  $\sigma_f^* = 0.2$  to  $0.9$  with correlation coefficients of  $r^2 \geq 0.9975$ .

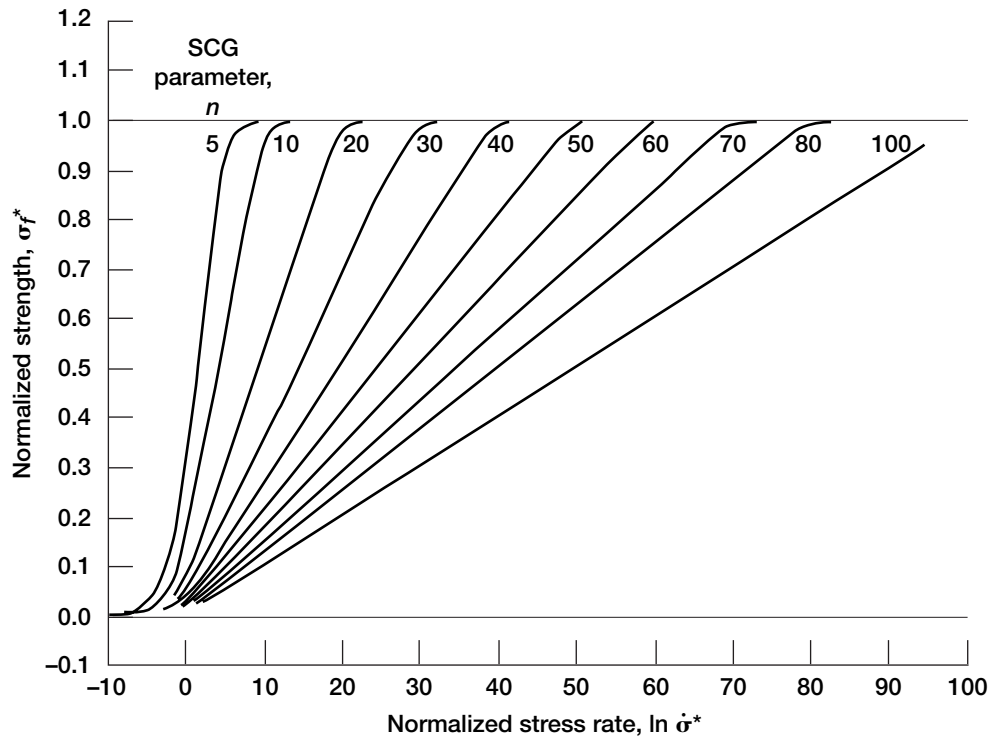


Figure 1.—Numerical results of normalized strength  $\sigma_f^*$  as a function of normalized stress rate  $\dot{\sigma}^*$  in constant stress rate loading for different values of SCG parameter  $n$ .

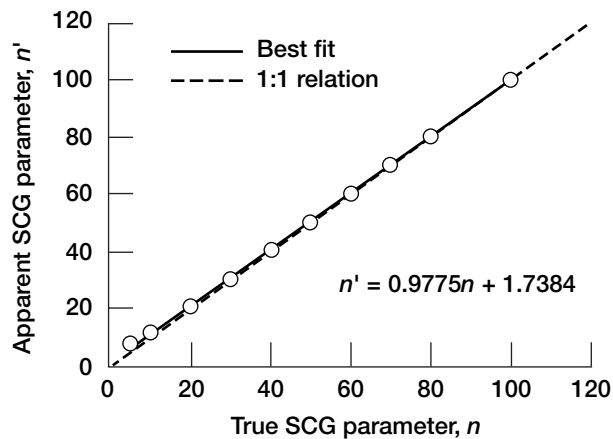


Figure 2.—Relationship between true SCG parameter  $n$  and apparent SCG parameter  $n'$  in constant stress rate loading.

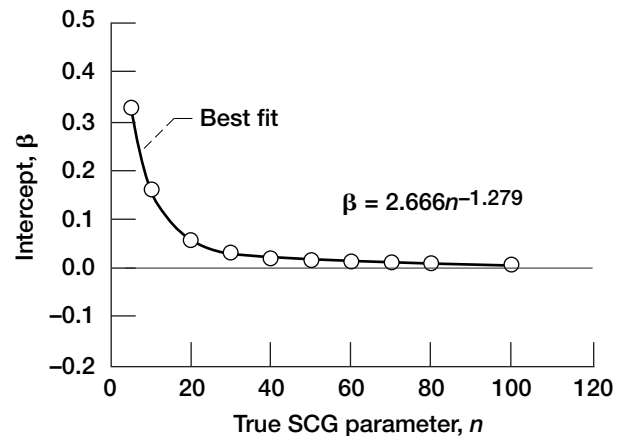


Figure 3.—Relationship between intercept  $\beta$  and true SCG parameter  $n$  in constant stress rate loading.

A linear regression analysis of  $\sigma_f^*$  and  $\ln \dot{\sigma}^*$  in the range of  $\sigma_f^* = 0.2$  to  $0.9$  was made to determine the slope and intercept of each individual curve for a given  $n$ , based on the following relation:

$$\sigma_f^* = \frac{1}{n'} \ln \dot{\sigma}^* + \beta \quad (18)$$

where  $1/n'$  and  $\beta$  are the slope and intercept, respectively. A comparison of the true  $n$  (an input datum) and the apparent  $n'$  (calculated) is shown in figure 2, where a linear relationship between  $n$  and  $n'$  is evident (except for lower  $n$  values, particularly for  $n < 10$ ). Hence, the overall relationship between  $n$  and  $n'$  can be approximated as

$$n' = 0.9775n + 1.7384 \quad (19)$$

with a correlation coefficient  $r^2 = 0.9995$ . Since the difference between  $n'$  and  $n$  was  $\geq 8$  percent for  $n \leq 10$  and  $\leq 3$  percent for  $n \geq 20$ , a further approximation of equation (19) can be made for  $n \geq 20$  as follows:

$$n' \approx n \quad (20)$$

The relationship between the intercept  $\beta$  and  $n$  is shown in figure 3. The value of  $\beta$  decreases with increasing  $n$  and becomes insignificant, approaching zero, when  $n > 20$ . The overall relationship between  $\beta$  and  $n$  in the range of  $n = 5$  to  $100$  was

$$\beta = 2.666(n)^{-1.279} \quad (21)$$

with a correlation coefficient  $r^2 = 0.9973$ . For a nonnormalized expression, equation (13) is used to reduce equation (18) to

$$\frac{\sigma_f}{S_i} = \frac{1}{n'} \ln \dot{\sigma} + \chi \quad (22)$$

where

$$\chi = \frac{1}{n'} \ln \left( \frac{a_i}{AS_i} \right) + \beta \quad (23)$$

SCG parameters  $n'$  and  $\chi$  in constant stress rate loading can be obtained from the slope and intercept by a linear regression analysis of  $(\sigma_f/S_i)$  versus  $\ln \dot{\sigma}$ . With  $n'$  thus calculated,  $n$  can be evaluated from equation (19). The parameter  $A$  can be evaluated using equation (23) from calculated  $\chi$  together with  $\beta$  (eq. (21)) and known values of  $a_i$  and  $S_i$ . The solution presented in this study (eq. (22)) is much simpler compared with the previous solution by Trantina (ref. 12) in which a simple linear regression would hardly be applicable because of the complex functional form of the solution, as shown below (note that  $\sigma_f$  is present in both sides of the equation):

$$\frac{\sigma_f}{S_i} = \frac{1}{n} \ln \dot{\sigma} + \frac{1}{n} \ln \left[ \left( \frac{2a_i}{A} \right) \sigma_f \right] \quad (24)$$

Therefore, when determining SCG parameters, the current solution (eq. (22)) significantly eliminates the complexity associated with a regression analysis that would be encountered in the previous solution (eq. (24)). For the case of  $n > 20$ , based on the results of figures 1 and 3 for  $\sigma_f^* \geq 0.4$ ,  $\beta$  is negligible (with a maximum of about 7 percent) compared with  $\sigma_f^*$ . Hence, equation (21) reduces to

$$\beta \approx 0 \quad (25)$$

which results in  $\chi \approx [\ln(a_i/AS_i)]/n$ . Likewise, in this case,  $n' \approx n$  from equation (20).

A distinct difference in functional expression between the power-law and exponential formulations is that in the power-law formulation,  $\log \sigma_f$  is plotted as a function of  $\log \dot{\sigma}$ , whereas in the exponential formulation,  $\sigma_f/S_i$  is plotted as a function of  $\ln \dot{\sigma}$  (fig. 1 shows this for the normalized parameters,  $\sigma_f^*$  as a function of  $\ln \dot{\sigma}^*$ ). Hence, the knowledge of inert strength in the exponential formulation is a prerequisite to determining SCG parameters  $n'$  and  $\chi$ , which is a disadvantage compared with the case (eq. (2)) of the power-law formulation. Note that the power-law formulation does not require any prior knowledge of inert strength to determine SCG parameters  $n$  and  $D$ . Although not presented here, the numerical result was plotted as  $\log \sigma_f$  as a function of  $\log \dot{\sigma}$  for different  $n$  values in the same way that is used for the power-law formulation. The resulting plots, however, showed appreciable nonlinearity, which made linear least-squares fitting inapplicable in determining the related SCG parameters.

### Constant Stress Loading

The numerical results of normalized time to failure  $T_f^*$  as a function of normalized applied stress  $\sigma^*$  are shown in figure 4 for values of  $n$  ranging from 5 to 100. The general trend of the solution can be summarized in terms of (1) the convergence of  $\ln T_f^*$  close to zero with  $\sigma^* \rightarrow 0$ , (2) the increased SCG susceptibility with decreasing  $n$  values, and (3) the linearity between  $\ln T_f^*$  and  $\sigma^*$  in the range of  $\sigma^* = 0.2$  to 0.9. As a consequence, the relationship between normalized time to failure and normalized applied stress within the linear region can be written as

$$\ln T_f^* = -n'\sigma^* + \beta \quad (26)$$

The linearity between  $\ln T_f^*$  and  $\sigma^*$  is manifest when the correlation coefficient  $r^2 \geq 0.995$  for each curve is considered. Hence,  $n'$  and  $\beta$  can be determined with a reasonable accuracy by a linear regression analysis based on the results in equation (26). The relationship between  $n'$  and  $n$  is shown in figure 5 and has the following relation:

$$n' = 0.9827n + 3.3440 \quad (27)$$

with  $r^2 = 0.9997$ . The difference between  $n'$  and  $n$  was  $\geq 8$  percent for  $n \leq 30$  and  $\leq 5$  percent for  $n \geq 40$  so that a further approximation of equation (27) can be made for the case of  $n \geq 40$  as

$$n' \approx n \quad (28)$$

The function of  $\beta$  with respect to  $n$  is depicted in figure 6, where the intercept  $\beta$  decreases with increasing  $n$  values, resulting in the best-fit relation

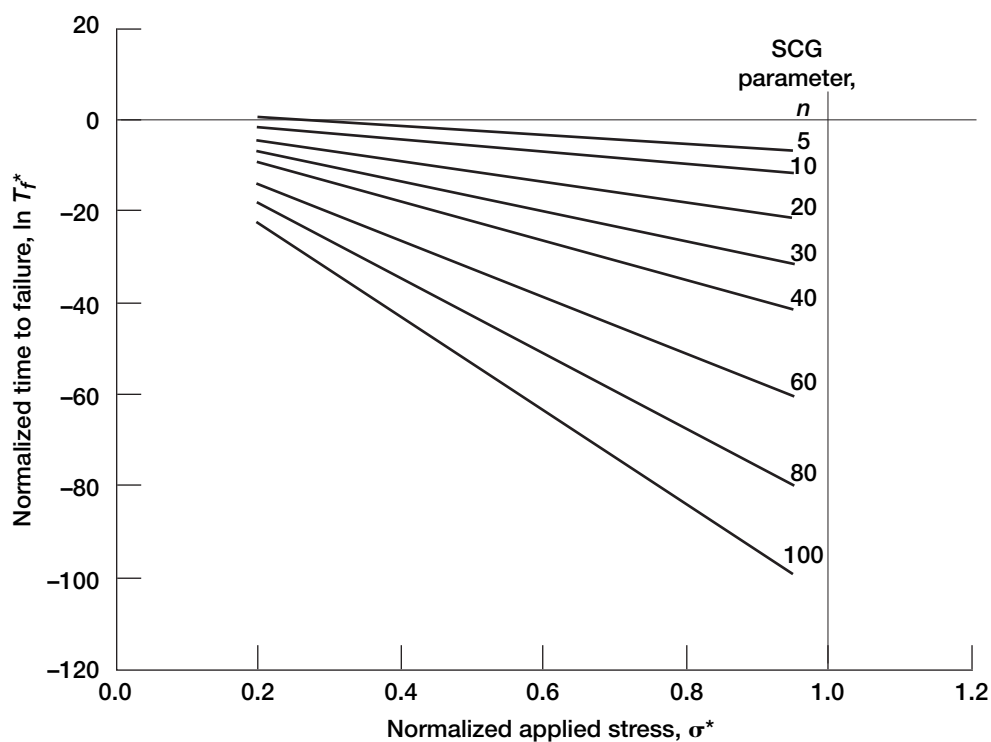


Figure 4.—Numerical results of normalized time to failure  $T_f^*$  as function of normalized applied stress  $\sigma^*$  in constant stress loading for different values of SCG parameter  $n$ .

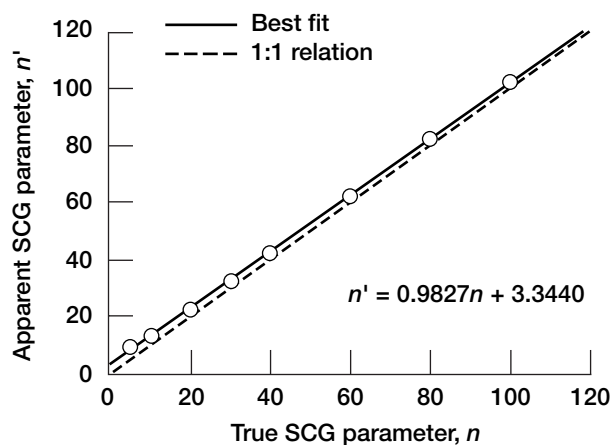


Figure 5.—Relationship between true SCG parameter  $n$  and apparent SCG parameter  $n'$  in constant stress loading.

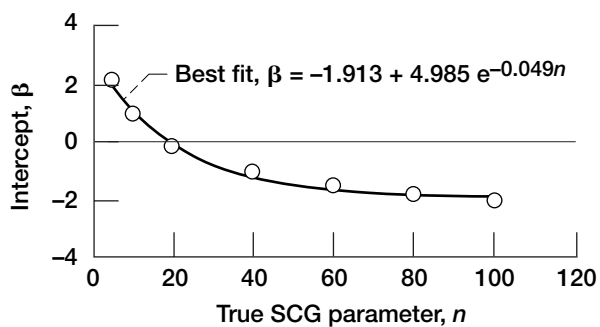


Figure 6.—Relationship between intercept  $\beta$  and true SCG parameter  $n$  in constant stress loading.

$$\beta = -1.913 + 4.985e^{-0.049n} \quad (29)$$

with  $r^2 = 0.9907$ .

For the nonnormalized expression, equation (13) is used to reduce equation (26) to

$$\ln t_f = -n' \frac{\sigma}{S_i} + \chi \quad (30)$$

where

$$\chi = \ln\left(\frac{a_i}{A}\right) + \beta \quad (31)$$

Hence,  $n'$  and  $\chi$  in constant stress loading can be obtained from the slope and intercept, respectively, by a simple linear regression analysis of the data  $\ln t_f$  as a function of  $\sigma/S_i$ . With  $n'$  calculated,  $n$  can be evaluated from equation (27). The parameter  $A$  can be estimated from equation (31) with calculated  $\chi$  together with  $\beta$  (eq. (29)) and known values of  $a_i$ . The solution presented here is much simpler than the rather complex one proposed by Trantina (ref. 12), in which the slope of the relation  $\ln t_f$  versus  $\sigma/S_i$  was  $n + (S_i/\sigma)$ . Both analyses, however, would give a similar result when  $n$  is significantly greater than  $S_i/\sigma$ .

A notable difference in constant stress loading analysis between the power-law and exponential formulations is that in the power-law formulation,  $\log t_f$  is plotted as a function of  $\log \sigma$  as seen in equation (3). However, in the exponential formulation,  $\ln t_f$  is plotted as a function of  $\sigma/S_i$ . Hence, as in the case of constant stress rate loading, inert strength must be known to determine  $n'$  and  $\chi$ , which is a distinctive drawback of the exponential formulation as compared with the power-law formulation. Unlike the power-law formulation (eq. (3)), there was significant nonlinearity in plots of  $\log T_f^*$  as a function of  $\log \sigma^*$  (not shown here), which made linear least-squares fitting inapplicable to determine the related SCG parameters.

### Cyclic Stress Loading

The results of the numerical solution of normalized time to failure  $T_f^*$  as a function of normalized maximum applied stress  $\sigma_{\max}^*$  in cyclic sinusoidal loading with two stress ratios  $R$  of 0.1 and 0.5 are shown in figure 7 for values of  $n$  ranging from 5 to 80. Similar to the case of constant stress loading, this plot is linear with respect to  $\sigma_{\max}^*$  in the range 0.2 to 0.9 and converges close to zero with a further decrease in  $\sigma_{\max}^*$ . The linearity between  $\ln T_f^*$  and  $\sigma_{\max}^*$  was evident considering the correlation coefficient of  $r^2 \geq 0.997$ . Also, note that the effect of the  $R$ -ratio on the solution for a given  $n$  value is insignificant. Based on the results in figure 7, similar to the case of constant stress loading, the relationship between  $\ln T_f^*$  and  $\sigma_{\max}^*$  can be described as follows:

$$\ln T_f^* = -n' \sigma_{\max}^* + \beta \quad (32)$$

where  $n'$  is the slope and  $\beta$  is the intercept, which can be determined from the numerical results using a linear regression analysis based on equation (32).

Figure 8 shows the relationship between  $n'$  and  $n$  for  $R$ -ratios of 0.1 and 0.5. For comparison, the result determined in constant stress loading (i.e.,  $R = 1.0$ ) from figure 5 was also included. Note that the SCG analysis in cyclic stress loading reduces to that of constant stress loading when  $R = 1.0$  (also seen in

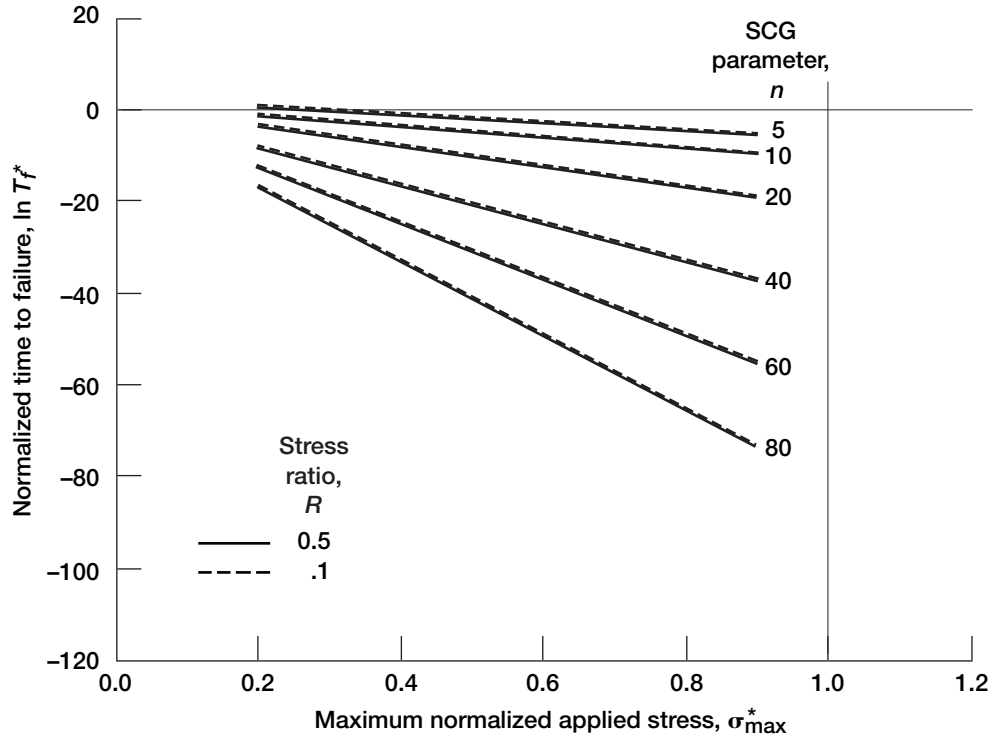


Figure 7.—Numerical results of normalized time to failure  $T_f^*$  as function of maximum normalized applied stress  $\sigma_{\max}^*$  in cyclic (sinusoidal) stress loading for  $R$ -ratios of 0.1 and 0.5 and different values of SCG parameter  $n$ .

eq. (17)); hence, constant stress loading can be regarded as one of the generalized cyclic loading configurations. A good linear relation between  $n'$  and  $n$  was found for both  $R = 0.1$  and  $0.5$  with  $r^2 > 0.999$ :

$$\begin{aligned} n' &= 0.9777n + 2.5296 \quad \text{for } R = 0.1 \\ n' &= 0.9772n + 2.5411 \quad \text{for } R = 0.5 \end{aligned} \quad (33)$$

As seen in figure 8, no significant difference in the  $n'$ - $n$  relation exists for  $R$ -ratios ranging from 0.1 to 1.0 (constant stress loading): this relationship is also observed in the power-law formulation (refs. 5, 17, and 18). The difference between  $n'$  and  $n$  was  $\geq 7$  percent for  $n \leq 20$  and  $\leq 3$  percent for  $n \geq 40$ , so that a further approximation of equation (33) can be made for  $R = 0.1$  and  $0.5$  for the case of  $n \geq 40$  as follows:

$$n' \approx n \quad (34)$$

The relationship between the intercept  $\beta$  and  $n$  is depicted in figure 9, where the result from figure 6 for constant stress loading ( $R = 1.0$ ) was also included for comparison. The intercepts for  $R = 0.1$  and  $0.5$  decrease more monotonically with increasing  $n$  values than that of  $R = 1.0$ . The best-fit equation, similar to equation (29) in constant stress loading, was obtained for each  $R$ -ratio:

$$\begin{aligned} \beta &= 0.1409 + 3.559e^{-0.0737n} \quad \text{for } R = 0.1 \\ \beta &= 0.1182 + 3.782e^{-0.0857n} \quad \text{for } R = 0.5 \end{aligned} \quad (35)$$

with the correlation coefficient of  $r^2 \geq 0.991$ .

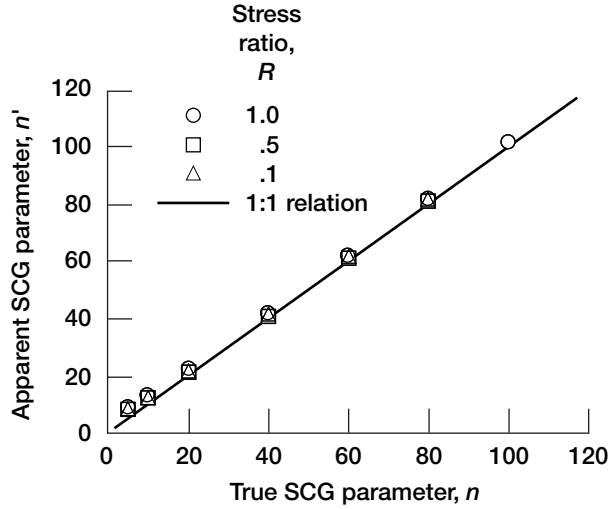


Figure 8.—Relationship between true SCG parameter  $n$  and apparent SCG parameter  $n'$  in cyclic (sinusoidal) stress loading with  $R$ -ratios of 0.1, 0.5, and 1.0.



Figure 9.—Relationship between intercept  $\beta$  and true SCG parameter  $n$  in cyclic (sinusoidal) stress loading with  $R$ -ratios of 0.1, 0.5, and 1.0.

For the nonnormalized expression, equation (13) can be used to reduce equation (32) to

$$\ln t_f = -n' \frac{\sigma_{\max}}{S_i} + \chi \quad (36)$$

where

$$\chi = \ln\left(\frac{a_i}{A}\right) + \beta \quad (37)$$

Therefore,  $n'$  and  $\chi$  of the exponential formulation in cyclic stress loading for a given  $R$ -ratio can be obtained from the slope and intercept by a linear regression analysis of the data of  $\ln t_f$  as a function of  $\sigma_{\max}/S_i$ . With  $n'$  calculated,  $n$  can be evaluated from equation (33). The SCG parameter  $A$  can be estimated from equation (37) with calculated  $\chi$ ,  $\beta$  (eq. (35)), and known values of  $a_i$ .

In cyclic stress loading, a distinct difference between the power-law and exponential formulations is that in the power-law formulation,  $\log t_f$  is plotted as a function of  $\log \sigma_{\max}$  as seen in equation (4) whereas in the exponential formulation,  $\ln t_f$  is plotted as a function of  $\sigma_{\max}/S_i$ . Hence, as in the cases of constant stress rate and constant stress loading, the inert strength of a material must be known in advance for cyclic loading to determine the corresponding SCG parameters, a clear disadvantage of the exponential formulation compared with the power-law formulation. Although not presented here, it was found that in plots of  $\log T_f^*$  as a function of  $\log \sigma_{\max}^*$ , typical of the power-law formulation (eq. (4)), there was a considerable nonlinearity that made linear least-squares fitting inapplicable in determining the related SCG parameters.

Typical examples of the effect of frequency on the time to failure in cyclic stress loading are shown in figure 10 for  $R$ -ratios of 0.1 and 0.5 with  $n = 20$  and  $\sigma_{\max}^* = 0.9$ . The number of cycles to failure  $N_f$  is calculated using the relation

$$N_f = \frac{T_f^*}{2\pi} \left( \frac{\omega a_i}{A} \right) \quad (38)$$



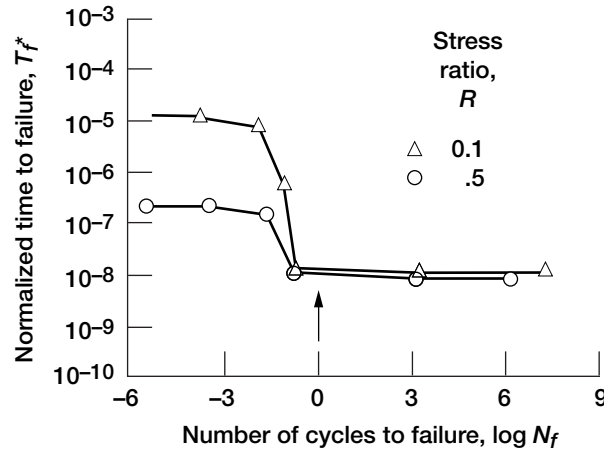


Figure 10.—Normalized time to failure  $T_f^*$  as function of number of cycles to failure  $N_f$  in cyclic (sinusoidal) stress loading for  $R$ -ratios of 0.1 and 0.5 under for  $n = 20$  and  $\sigma_{\max}^* = 0.9$ . Arrow indicates number of cycles corresponding to unity.

where  $\omega a_i/A$  is the input value for numerical procedures, as shown in equation (17). As seen from figure 10,  $T_f^*$  decreases rapidly around  $N_f = 10^{-1}$  to  $10^0$  and thereafter approaches a plateau that corresponds to the exact solution of time to failure. In other words, as long as the total number of cycles to failure is  $\geq 1$ , the solution of time to failure is converged and is thus independent of either the number of cycles or frequency. Hence in the numerical procedure, the value of  $\omega a_i/A$  was chosen to fulfill this requirement and started with a minimum value of  $10^8$ , depending on  $T_f^*$ . This frequency independency in the exponential formulation is the same as that in the power-law formulation (refs. 5 and 17). For the power-law formulation, the time to failure remains unchanged with frequency, which is attributed to the

functional form of  $(1/\tau) \int_0^\tau [f(t)]^n dt$  in equation (7) (ref. 5). It can be easily shown that the period  $\tau$  of

any applied loading cycle is always cancelled out in the formulation. It should be noted that in this analysis, the presence of another damage mechanism (refs. 19 to 21) in addition to slow crack growth was not assumed to occur in cyclic loading. Only slow crack growth was considered as the unique mechanism leading to the failure of a material.

### Comparison of Constant Stress and Cyclic Stress Loading Lifetimes: the $h$ -Ratio

The ratio of constant stress and cyclic stress loading lifetimes, the  $h$ -ratio, with a condition of  $\sigma^*$  in constant stress loading equal to  $\sigma_{\max}^*$  in cyclic stress loading ( $\sigma^* = \sigma_{\max}^*$ ), has been frequently used in the power-law formulation (refs. 5, 17, and 18) to quickly compare lifetimes of constant stress and cyclic stress loading. As done customarily for the power-law formulation, the  $h$ -ratio was also calculated for the exponential formulation. The  $h$ -ratio is defined as (refs. 5, 17, and 18)

$$h = \frac{t_{fs}}{t_{fc}} \quad (39)$$

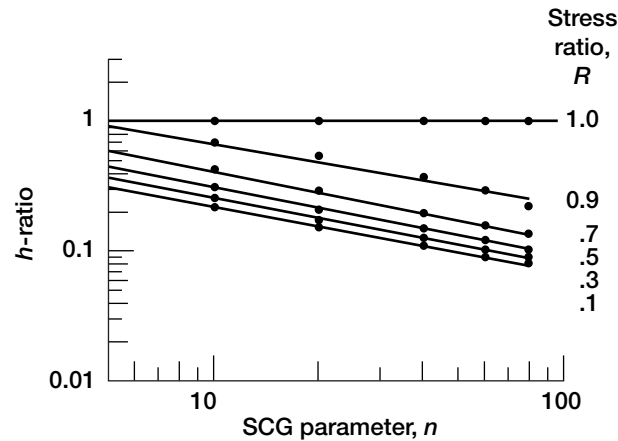


Figure 11.—Ratio of constant stress to cyclic stress loading lifetimes,  $h$ -ratio, as function of SCG parameter  $n$  for different  $R$ -ratios in cyclic (sinusoidal,  $\sigma_{\max}^* = 0.7$ ) loading. Each line represents best fit.

where again  $t_{fs}$  and  $t_{fc}$  are times to failure in constant stress and cyclic stress loading, respectively. Based on the numerical results of time to failure for constant stress and cyclic stress loading, the  $h$ -ratio was calculated using equation (39), and the results are presented in figure 11 as a function of  $n$  for different values of the  $R$ -ratio. The  $h$ -ratio decreases with increasing  $n$ , and the rate of decrease with increasing  $n$  is almost the same regardless of  $R$ -ratio up to 0.9. Also, for a given  $n$ , the  $h$ -ratio increases with increasing  $R$ -ratio. Note that the  $R$ -ratio of 1.0 represents the case for constant stress loading so that when  $R = 1.0$ , the numerical solution in cyclic stress loading should reduce to the case of constant stress loading. This presents another way to check the accuracy of the cyclic stress loading analysis. The  $h$ -ratio varies slightly by a factor of 2 between  $\sigma_{\max}^* = 0.2$  and 0.9: the lower  $\sigma_{\max}^*$  value gives the higher  $h$ -ratio and vice versa. Hence, for a conservative estimate, the higher value of  $\sigma_{\max}^* = 0.7$  was used for the calculation of the  $h$ -ratio. The maximum difference in life between cyclic stress and constant stress loading is approximately 1 order of magnitude, which occurs for  $R = 0.1$  and  $n \geq 80$ . A similar trend in the  $h$ -ratio can also be observed in the power-law formulation, as shown in figure 12. Here the plots of the  $h$ -ratio as a function of  $n$  are shown for a range of  $R = 0.0$  to 1.0, calculated previously for the power-law formulation (ref. 17). Unlike the exponential formulation, no effect of  $\sigma_{\max}^*$  on the  $h$ -ratio for a given  $R$ -ratio had been observed for the power-law formulation. Although no significant difference exists in the plots of  $h$ -ratio versus  $n$  between the exponential (fig. 11) and power-law (fig. 12) formulations, the overall magnitude of the  $h$ -ratio is about 20 percent greater in the exponential than in the power-law formulation.

### Other Exponential Formulations

A comparison of solutions from other exponential SCG formulations under three loading configurations is shown in figure 13. The figure presents the results of three exponential formulations of equations (9) to (11) for  $n = 20$  and 40 and compares them with those of the primary formulation of equation (8). The difference in solution between this primary equation and two other equations ((9) and (10)) was insignificant, particularly at higher stress rates (in constant stress rate loading, fig. 13(a)) and higher applied stresses (in constant stress and cyclic stress loading, figs. 13(b) and (c)), giving rise to a reasonable linearity between the dependent/independent variables related. This insignificant difference in solution as well as the linearity allows one to conclude that the primary equation would be representative of all three exponential SCG formulations considered. By contrast, the remaining second-order formulation

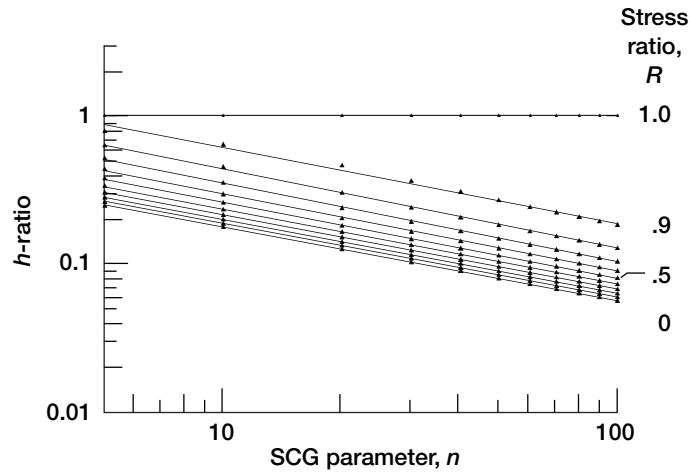


Figure 12.—Ratio of constant stress to cyclic stress loading lifetimes,  $h$ -ratio, as function of SCG parameter  $n$  for different  $R$ -ratios in cyclic (sinusoidal) stress loading, calculated with power-law formulation (ref. 17). Each line represents best fit.

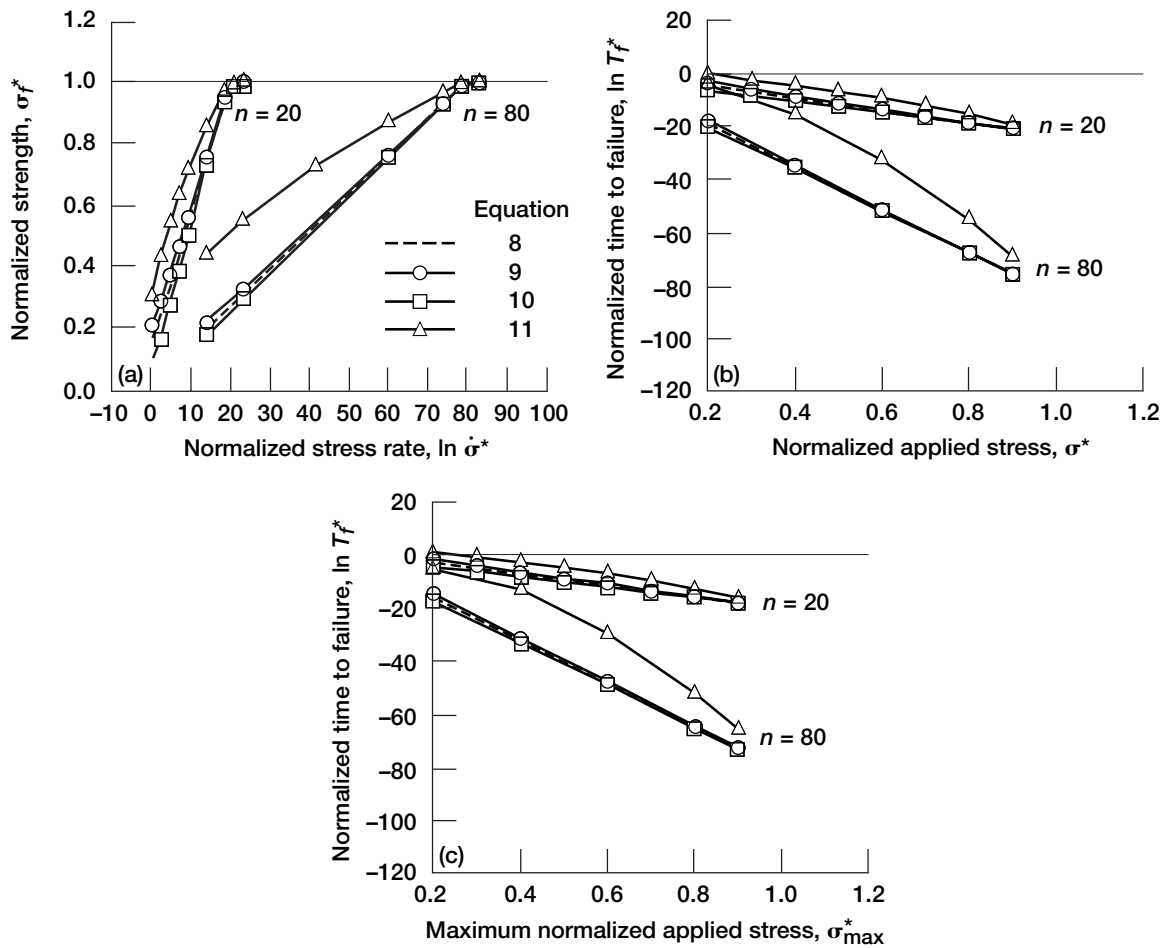


Figure 13.—Results of numerical solutions using three exponential formulations of equations (9) to (11) compared with the primary exponential formulation of equation (8) for selected SCG parameters of  $n = 20$  and  $80$ . (a) Constant stress-rate loading. (b) Constant stress loading. (c) Cyclic stress loading with stress ratio  $R = 0.1$ .

(eq. (11)) showed an appreciable deviation and a notable nonlinearity. Therefore, the determination of corresponding SCG parameters in this case differs from that of the primary equation and should only be attempted under appropriate circumstances so that a simple linear regression could be applied with reasonable accuracy. It is expected that considering its functional form, equation (12) also would yield results similar to those of equation (11).

Although the use of the exponential formulations to determine the SCG parameters of a material requires somewhat inconvenient numerical procedures, the resulting solutions given in this report would have almost the same degree of simplicity in both data analysis and experiments as the power-law formulation in constant stress rate, constant stress, or cyclic stress loading configurations. However, the knowledge of inert strength of a material should be known beforehand so that the corresponding SCG parameters (particularly  $n$ ) can be determined, which could be a major drawback of the exponential formulation. In parts 2 and 3 of this report series, a variety of experimental data from various glasses and advanced ceramics at both ambient and elevated temperatures will be used to verify the solutions given herein.

## Conclusions

Based on the numerical solutions of life prediction parameters obtained with exponential formulations, the following conclusions were made:

1. In constant stress rate (dynamic fatigue) loading, slow-crack-growth (SCG) parameters can be determined by a linear regression analysis of the data of (fracture stress/inert strength) as a function of applied stress rate,  $\sigma_f/S_i$  versus  $\ln \dot{\sigma}$ , together with the appropriate relations provided.
2. In constant stress (static fatigue or stress rupture) and cyclic stress (cyclic fatigue) loading, the corresponding SCG parameters can be evaluated by a linear regression analysis of the data of time to failure as a function of (maximum applied stress/inert strength),  $\ln t_f$  versus  $\sigma_{\max}/S_i$ , in conjunction with the pertinent relations provided.
3. No frequency effect on life and no dependency of SCG parameter  $n$  on the  $R$ -ratio, the ratio of minimum to maximum applied stress, was observed in cyclic stress loading, much the same as that observed for the power-law formulation. The difference in the ratio of constant stress to cyclic stress loading lifetimes, the  $h$ -ratio, between the exponential and power-law formulations was minimal with a maximum difference of about 20 percent.
4. While the numerical solutions using the exponential formulation require somewhat inconvenient numerical procedures, they provide almost the same level of simplicity in both data analysis and experiments as the power-law formulation. However, requiring the knowledge of the inert strength of a material to determine corresponding SCG parameters (particularly  $n$ ) would make the exponential formulation more difficult to use in comparison with the power-law formulation.
5. There is no appreciable difference in solutions between the primary exponential SCG equation (used in this analysis) and two other exponential expressions analyzed in this study, so the primary equation would be considered representative of all the first-order exponential formulations.

## Appendix—Symbols

$A$	slow-crack-growth parameter defined in equations (1) and (8)
$a$	crack size
$B$	slow-crack-growth parameter, $B = 2K_{IC} / [AY^2(n - 2)]$
$C$	crack size in normalized scheme of references 15 to 17
$D$	slow-crack-growth parameter defined in equations (5) to (7)
$f(t)$	periodic function, cyclic loading
$h$	ratio of constant to cyclic stress loading lifetimes
$K$	stress intensity factor
$N$	number of cycles
$n$	slow-crack-growth parameter defined in equations (1) and (8)
$R$	stress ratio
$r^2$	correlation coefficient
$S$	strength
$T$	time in normalized scheme of references 15 to 17
$t$	time
$v$	crack velocity
$Y$	crack geometry factor
$\beta$	intercept of curve in linear regression analysis defined in equations (18), (26), and (32)
$\chi$	slow-crack-growth parameter defined in equations (23), (31), and (37)
$\sigma$	applied stress
$\dot{\sigma}$	applied stress rate
$\tau$	period
$\omega$	angular velocity

Subscripts:

$C$	critical
$c$	cyclic stress
$d$	constant stress rate
$f$	fracture
$I$	mode I
$i$	inert or initial condition
max	maximum
min	minimum
$s$	constant stress

Superscripts:

*	normalized
'	apparent (calculated)

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